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Integrable Systems and W-algebras

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Abstract

The basic concepts underlying our analysis of *W-algebras* as extended symmetries of integrable systems are summarized. The construction starts from the second hamiltonian structure of "Generalized Drinfel'd-Sokolov" hierarchies, and its correspondence with the A_1-embeddings is established, providing a rather simple and general scheme.

We summarize the building blocks of our results[1] about the construction and classification of *W-algebras*[2] starting from the "Generalized Drinfel'd-Sokolov" (G.D-S) integrable non-linear hierarchies[3], and their relation to the method of hamiltonian reduction and A_1-embeddings[4].

Classical *W-algebras* are non-linear chiral extensions of the conformal Virasoro algebra generated by primary fields $w_i(z)$:

$$\{w_i(z_1), w_j(z_2)\} \quad = \quad \sum_k P_{ij}^k(w_1, ...w_N)\ \delta^{(k)}(z_1 - z_2)\,, \tag{1}$$

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where $\{,\}$ is a Poisson bracket, and P^k_{ij} are differential polynomials constrained by the Jacobi identity. The energy-momentum tensor is assigned to $i = 1$

$$\{T(z_1), w_i(z_2)\} = \; - \Delta_i \, w_i(z_1) \, \delta'(z_1 - z_2) \, - (\Delta_i - 1) \, w'_i(z_1) \, \delta(z_1 - z_2) \quad (2)$$

and Δ_i is the conformal weight of $w_i(z)$.

W-algebras appear in many different applications of two-dimensional classical and quantum field theory, such as String Theory, Gravity, Critical Phenomena, or Solid State, and offer good prospects for combining external and internal symmetries in theories of higher spins. *W-algebras* generally describe symmetries of two-dimensional integrable systems. In particular, they correspond to the "second" hamiltonian structure of G.D-S hierarchies, which can be related in a simple way to the general construction based on A_1-embeddings.

We remind that the second hamiltonian structure of the K.d.V. hierarchies associated to the affine Kac-Moody algebras $A^{(1)}_n$ [5] lead to the well known W_{n+1} algebras, the first of which is, of course, the Virasoro algebra:

$$\{u(x,t), u(y,t)\}_2 = \frac{1}{2} \, \delta'''(x-y) - 2u(x,t) \, \delta'(x-y) - u'(x,t) \, \delta(x-y) \; , \quad (3)$$

which corresponds to $n = 1$ and, hence, to the original K.d.V. equation

$$\begin{aligned} \partial_t u(x,t) \;\; &= \;\; - \frac{1}{4} \, u'''(x,t) \, + \, \frac{3}{2} \, u(x,t) u'(x,t) \\ &= \;\; \{ \int dx \frac{u^2(x,t)}{4}, u(y,t)\}_2 \; . \end{aligned} \quad (4)$$

In general, the second Poisson-bracket of the G.D-S hierarchies provides different *W-algebras* associated to distinguished elements of every affine algebra, and the conformal structure arises when the well known scale invariance of the equations is made local. It is convenient to explain one of the basic ingredients of the construction: the Heisenberg subalgebras (H.S.A) of Kac-Moody algebras, which, in practice, will be considered just as loop algebras,

$$\hat{g}^{(1)} \; = \; \mathbf{C}[z, z^{-1}] \bigotimes \, g \, \bigoplus \mathbf{C} \, d \, , \quad (5)$$

since the central extension $\mathbf{C}c$ can and will be ignored. g is the underlying finite Lie algebra, and $\mathbf{C}[z, z^{-1}]$ are the Laurent polynomials in the affine

parameter z. The standard derivation $d = z\frac{d}{dz}$ induces a gradation of $\hat{g}^{(1)}$, which generally means a decomposition:

$$\hat{g}^{(1)} = \bigoplus_j \hat{g}_j^{(1)}\,, \quad [\hat{g}_j^{(1)}, \hat{g}_k^{(1)}] \subset \hat{g}_{j+k}^{(1)}. \tag{6}$$

Notice that $[d, \hat{g}_j^{(1)}] = j\,\hat{g}_j^{(1)}$ if $\hat{g}_j^{(1)} = z^j \otimes g$ and, so, this distinguished gradation is called *homogeneous*. Also if H is a Cartan subalgebra, $z^j \otimes H$ is a maximal commuting subalgebra: the *homogeneous* example of the **Heisenberg subalgebras**. General H.S.A's correspond to all the possible generalizations of such structure, which are in one to one relation with the conjugacy classes of the Weyl group of g.

For each conjugacy class $[w]$ there is an associated set of $r+1$ (r is the rank of g) coprime non-negative integers called $\vec{s}_w$, which will specify the grades under the new derivation $d_{s_w} = N_s d + H_{s_w}$, with $N_{s_w} = \sum k_i s_w^i$ (k_i being the Kac labels, i.e., the components of the right null eigenvector of the extended Cartan matrix), and $\alpha_i(H_{s_w}) = s_w^i$. The way to prove that this actually generates all the inequivalent Heisenberg subalgebras, which will be labelled as $\mathcal{H}[w]$, is to express an inner automorphism of g associated to $[w]$ as an imaginary-phase operator specified by the adjoint action of H_{s_w}, which can be expressed as $\sum_i \frac{2}{{\alpha_i}^2} s_w^i w_i H$, with w_i being the fundamental weights[1]. The *homogeneous* case corresponds to the identity $w = 1$ and $\vec{s}_w = (1, 0, ..., 0)$, while the product of all the Weyl reflections, i.e., the Coxeter element w_c, corresponds to the *principal* gradation. For $sl(2, \mathbf{C})$ these are the only cases, but, for instance, in A_2 there is also the $\vec{s}_w = (2, 1, 1)$, which yields the Heisenberg subalgebra generated by:

$$b_{4m} = z^m \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} \; ; \qquad b_{2+4m} = z^m \begin{pmatrix} & & 1 \\ & 0 & \\ z & & \end{pmatrix} \; ; \qquad m \in \mathbf{Z}. \tag{7}$$

The name of these algebras comes from their central extension ignored here, where the different elements X_p , $[d_{s_w}, X_p] = pX_p$, would satisfy:

$$[X_p, X_q] = p\,\delta_{p+q,0}\; c\,. \tag{8}$$

Notice that they are maximally commuting for $c = 0$, and the suggestive similarity of d_{s_w} with the momentum. In fact the former relations can be seen as the (2-dimensional) Poincaré algebra.

We are now in position to summarize the G.D-S construction of non-linear differential hierarchies. A Lax operator is then defined for every constant element of positive s_w-grade, $\Lambda^i \in \mathcal{H}[w]$,

$$L \, = \, \partial_x + \Lambda^i(z) + q(x,z) \, , \qquad q(x,z) \in Q^{<i}_{\geq 0} \equiv \hat{g}_{\geq 0}(s) \bigcap \hat{g}_{<i}(s_w) \tag{9}$$

above, s is some other gradation of $\hat{g}$ such that $s \leq s_w$. The partial ordering implies[3] that $\hat{g}_{j(s_w)} \subset \hat{g}_{j(s)}$, for $j \geq 0$, $j \leq 0$ and $j = 0$.

Q, the phase space the of hierarchy, is invariant under the *gauge transformations* generated by the adjoint action of $S = \hat{g}_0^{<0}$. Moreover, as long as $Ker(ad\Lambda) = \mathcal{H}[w]$ †, one can "diagonalize" L in $\mathcal{H}[w]$ by means of an element $V \in Im(ad\Lambda) \bigcap \hat{g}^{<0}$:

$$\mathcal{L} \, = e^{adV} \, (L) \, = \, \partial_x + \Lambda^i(z) + \sum_{j<i} H_j(x) \tag{10}$$

Since $[M_+, L] \, = \, -[M_-, L] \, \in \, C^\infty(\mathbf{R}, Q)$ for M commuting with L, one can define the flows:

$$\frac{\partial L}{\partial t_{b_j}} \, = [(exp(-adV)(b_j))^{\geq 0_{s_w}}, L] = \, - \, [(exp(-adV)(b_j))^{<0_{s_w}}, L] \, , \tag{11}$$

$b_j \in \mathcal{H}[w]$, or through similar equations involving of the lower gradation s.

Provided that $Ker(ad\Lambda) \bigcap S = \emptyset$ (injective gauge transformations), the non-linear differential equations for the *(canonical)* gauge invariant functionals q^{can} will be polynomial, as required by locality in the field theory applications, then, Q^{can} will be just a complementary space of $[\Lambda, S]$. For regular elements, $Ker(ad\Lambda) \bigcap S = \emptyset$ is always satisfied (our[1] Lemma 1).

By construction, the H_j for $j \, < \, 0$ are the conserved densities of the hierarchy (for $i > j \geq 0$ they are just centers), the hierarchy can be written in zero curvature form, and more important, it is bihamiltonian and the second Poisson bracket exhibits conformal invariance.

It is then natural to investigate whether such structures are related to the W-algebras obtained by hamiltonian reduction of the affine Kac-Moody current algebras. Moreover, the latter are classified by the A_1-embeddings, which are subalgebras $\{J_\pm, J_0\}$ of g such that, under their ad-action, g decomposes into (spin) irreducible representations of that $sl(2,\mathbf{C})$. Their J_+

†A condition which can be relaxed for non-regular elements of $\mathcal{H}[w]$.

are related to the reduced currents J^{red} by addition of minimal weights of the spin decomposition. In A_n the embeddings are in correspondence with the Weyl group, and they are also labeled by partitions of n. So, one expects them to be related to the hierarchies arising from Heisenberg subalgebras. The precise way in which this happens is one of the results of our paper[1]. The first step is to derive a generator of the conformal transformations that generalize the local scale invariance of the hierarchy, $q^k \to \lambda^{k/i-1} q^k$ when $x \to \lambda x$ and $t_{b^j} \to \lambda^{j/i} t_{b^j}$. Such an energy momentum tensor is easily derived upon restriction to the functionals of zero lower grade $q_0(x)$ since, for them, the second Poisson bracket simplifies to the Kirillov bracket

$$\{\varphi, \psi\}_2 \;=\; \Big([(d_q\varphi)_0, (d_q\psi)_0], (\Lambda^i + q(x))_0\Big) - ((d_q\varphi)_0, \partial_x (d_q\psi)_0)\ , \quad (12)$$

where $(,)$ is the generalization of the Killing form $<,>$ to the affine algebra

$$(A, B) \;=\; \sum_{k\in\mathbf{Z}} \int dx < A_k(x), B_{-k}(x) > \quad (13)$$

and d_q is the Frechet derivative.

Restricting further the construction to the case when $Ker(ad\Lambda_0) \bigcap S\ = \emptyset$, one shows directly that the second bracket is a W-algebra. A previous step is to show, by a counting argument, that the choice of q^{can} is such that the restriction of the positive grade components $(q^{can>0}(x))_0 = 0$, and that those of $q^{can\,\le 0}(x)$ depend only on $q_0(x)$. The embedding corresponding to the W-algebra associated to $\{\hat{g}, \Lambda^{i>0} \in \mathcal{H}[w], s_w, s\}$ is specified by $J_+ = (\Lambda)_0$. The reverse of this result[1], Theorem 2, is the answer to the question of which J_+ are obtained in this way from the corresponding Heisenberg subalgebra. The solution turns out quite restrictive, and it is given[1] by Theorem 3: for A_n, the *W-algebras* that can be constructed from Heisenberg subalgebras with the G.D-S formalism are just those corresponding to the embeddings labelled by partitions of the form $n+1 = k(m) + q(1)$ or $n+1 = k(m+1) + k(m) + q(1)$, where k and q indicate the times the integer in the bracket is repeated.

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